pressure for a cambered airfoil. However, note that A_1 and A_2 depend only on the shape of the camber line and do not involve the angle of attack. Hence, from Eq. (4.64), $c_{m,c/4}$ is *independent* of α. Thus, the quarter-chord point is the *theoretical location* of the aerodynamic center for a cambered airfoil.

The location of the center of pressure can be obtained from Eq. (1.21):

$$x_{cp} = -\frac{M'_{LE}}{L'} = -\frac{c_{m,le}c}{c_l} \tag{4.65}$$

Substituting Eq. (4.63) into (4.65), we obtain

$$\boxed{x_{cp} = \frac{c}{4}\left[1 + \frac{\pi}{c_l}(A_1 - A_2)\right]} \tag{4.66}$$

Equation (4.66) demonstrates that the center of pressure for a cambered airfoil varies with the lift coefficient. Hence, as the angle of attack changes, the center of pressure also changes. Indeed, as the lift approaches zero, x_{cp} moves toward infinity; i.e., it leaves the airfoil. For this reason, the center of pressure is not always a convenient point at which to draw the force system on an airfoil. Rather, the force-and-moment system on an airfoil is more conveniently considered at the aerodynamic center. (Return to Fig. 1.19 and the discussion at the end of Sec. 1.6 for the referencing of the force-and-moment system on an airfoil.)

Example 4.2. Consider an NACA 23012 airfoil. The mean camber line for this airfoil is given by

$$\frac{z}{c} = 2.6595\left[\left(\frac{x}{c}\right)^3 - 0.6075\left(\frac{x}{c}\right)^2 + 0.1147\left(\frac{x}{c}\right)\right] \qquad \text{for } 0 \le \frac{x}{c} \le 0.2025$$

and

$$\frac{z}{c} = 0.02208\left(1 - \frac{x}{c}\right) \qquad \text{for } 0.2025 \le \frac{x}{c} \le 1.0$$

Calculate (*a*) the angle of attack at zero lift, (*b*) the lift coefficient when $\alpha = 4°$, (*c*) the moment coefficient about the quarter chord, and (*d*) the location of the center of pressure in terms of x_{cp}/c, when $\alpha = 4°$. Compare the results with experimental data.

Solution. We will need dz/dx. From the given shape of the mean camber line, this is

$$\frac{dz}{dx} = 2.6595\left[3\left(\frac{x}{c}\right)^2 - 1.215\left(\frac{x}{c}\right) + 0.1147\right] \qquad \text{for } 0 \le \frac{x}{c} \le 0.2025$$

and

$$\frac{dz}{dx} = -0.02208 \qquad \text{for } 0.2025 \le \frac{x}{c} \le 1.0$$

Transforming from x to θ, where $x = (c/2)(1 - \cos\theta)$, we have

$$\frac{dz}{dx} = 2.6595\left[\frac{3}{4}(1 - 2\cos\theta + \cos^2\theta) - 0.6075(1 - \cos\theta) + 0.1147\right]$$

or

$$= 0.6840 - 2.3736\cos\theta + 1.995\cos^2\theta \qquad \text{for } 0 \le \theta \le 0.9335 \text{ rad}$$

and

$$= -0.02208 \qquad \text{for } 0.9335 \le \theta \le \pi$$

(*a*) From Eq. (4.61),

$$\alpha_{L=0} = -\frac{1}{\pi}\int_0^\pi \frac{dz}{dx}(\cos\theta - 1)\,d\theta$$

(*Note*: For simplicity, we have dropped the subscript zero from θ; in Eq. (4.61), θ_0 is the variable of integration—it can just as well be symbolized as θ for the variable of integration.) Substituting the equation for dz/dx into Eq. (4.61), we have

$$\alpha_{L=0} = -\frac{1}{\pi}\int_0^{0.9335}(-0.6840 + 3.0576\cos\theta - 4.3686\cos^2\theta + 1.995\cos^3\theta)\,d\theta$$

$$-\frac{1}{\pi}\int_{0.9335}^{\pi}(0.02208 - 0.02208\cos\theta)\,d\theta \tag{E.1}$$

From a table of integrals, we see that

$$\int \cos\theta\,d\theta = \sin\theta$$

$$\int \cos^2\theta\,d\theta = \tfrac{1}{2}\sin\theta\cos\theta + \tfrac{1}{2}\theta$$

$$\int \cos^3\theta\,d\theta = \tfrac{1}{3}\sin\theta(\cos^2\theta + 2)$$

Hence, Eq. (E.1) becomes

$$\alpha_{L=0} = -\frac{1}{\pi}[-2.8683\theta + 3.0576\sin\theta - 2.1843\sin\theta\cos\theta$$

$$+ 0.665\sin\theta(\cos^2\theta + 2)]_0^{0.9335}$$

$$-\frac{1}{\pi}[0.02208\theta - 0.02208\sin\theta]_{0.9335}^{\pi}$$

Hence,

$$\alpha_{L=0} = -\frac{1}{\pi}(-0.0065 + 0.0665) = -0.0191 \text{ rad}$$

or

$$\boxed{\alpha_{L=0} = -1.09°}$$

(*b*) $\alpha = 4° = 0.0698$ rad
From Eq. (4.60),

$$c_l = 2\pi(\alpha - \alpha_{L=0}) = 2\pi(0.0698 + 0.0191) = \boxed{0.559}$$

(*c*) The value of $c_{m,c/4}$ is obtained from Eq. (4.64). For this, we need the two Fourier coefficients, A_1 and A_2. From Eq. (4.51),

$$A_1 = \frac{2}{\pi}\int_0^\pi \frac{dz}{dx}\cos\theta\,d\theta$$

$$A_1 = \frac{2}{\pi}\int_0^{0.9335} (0.6840 \cos\theta - 2.3736 \cos^2\theta + 1.995 \cos^3\theta)\, d\theta$$

$$+\frac{2}{\pi}\int_{0.9335}^{\pi} (-0.02208 \cos\theta)\, d\theta$$

$$= \frac{2}{\pi}[0.6840 \sin\theta - 1.1868 \sin\theta \cos\theta - 1.1868\theta + 0.665 \sin\theta(\cos^2\theta + 2)]_0^{0.9335}$$

$$+\frac{2}{\pi}[-0.02208 \sin\theta]_{0.9335}^{\pi}$$

$$= \frac{2}{\pi}(0.1322 + 0.0177) = 0.0954$$

From Eq. (4.51),

$$A_2 = \frac{2}{\pi}\int_0^{\pi} \frac{dz}{dx}\cos 2\theta\, d\theta = \frac{2}{\pi}\int_0^{\pi}\frac{dz}{dx}(2\cos^2\theta - 1)\, d\theta$$

$$= \frac{2}{\pi}\int_0^{0.9335} (-0.6840 + 2.3736 \cos\theta - 0.627 \cos^2\theta$$

$$-4.747 \cos^3\theta + 3.99 \cos^4\theta)\, d\theta$$

$$+\frac{2}{\pi}\int_{0.9335}^{\pi} (0.02208 - 0.0446 \cos^2\theta)\, d\theta$$

Note:

$$\int \cos^4\theta\, d\theta = \frac{1}{4}\cos^3\theta \sin\theta + \frac{3}{8}(\sin\theta\cos\theta + \theta)$$

Thus,

$$A_2 = \frac{2}{\pi}\left\{-0.6840\theta + 2.3736 \sin\theta - 0.628\left(\frac{1}{2}\right)(\sin\theta\cos\theta + \theta)\right.$$

$$\left.-4.747\left(\frac{1}{3}\right)\sin\theta(\cos^2\theta + 2) + 3.99\left[\frac{1}{4}\cos^3\sin\theta + \frac{3}{8}(\sin\theta\cos\theta + \theta)\right]\right\}_0^{0.9335}$$

$$+\frac{2}{\pi}\left[0.02208\theta - 0.0446\left(\frac{1}{2}\right)(\sin\theta\cos\theta + \theta)\right]_{0.9335}^{\pi}$$

$$= \frac{2}{\pi}(0.11384 + 0.01056) = 0.0792$$

From Eq. (4.64),

$$c_{m,c/4} = \frac{\pi}{4}(A_2 - A_1) = \frac{\pi}{4}(0.0792 - 0.0954)$$

$$\boxed{c_{m,c/4} = -0.0127}$$

(*d*) From Eq. (4.66),

$$x_{cp} = \frac{c}{4}\left[1 + \frac{\pi}{c_l}(A_1 + A_2)\right]$$

Hence,

$$\frac{x_{cp}}{c} = \frac{1}{4}\left[1 + \frac{\pi}{0.559}(0.0954 - 0.0792)\right] = 0.273$$

COMPARISON WITH EXPERIMENTAL DATA. The data for the NACA 23012 airfoil are shown in Fig. 4.22. From this, we make the following tabulation:

	Calculated	Experiment
$\alpha_{L=0}$	$-1.09°$	$-1.1°$
c_l (at $\alpha = 4°$)	0.559	0.55
$c_{m,c/4}$	-0.0127	-0.01

Note that the results from thin airfoil theory for a cambered airfoil agree very well with the experimental data. Recall that excellent agreement between thin airfoil theory for a symmetric airfoil and experimental data has already been shown in Fig. 4.20. Hence, all of the work we have done in this section to develop thin airfoil theory is certainly worth the effort. Moreover, this illustrates that the development of thin airfoil theory in the early 1900s was a crowning achievement in the theoretical aerodynamics and validates the mathematical approach of replacing the chord line of the airfoil with a vortex sheet, with the flow tangency condition evaluated along the mean camber line.

This brings to an end our introduction to classical thin airfoil theory. Returning to our road map in Fig. 4.2, we have now completed the right-hand branch.

4.9 LIFTING FLOWS OVER ARBITRARY BODIES: THE VORTEX PANEL NUMERICAL METHOD

The thin airfoil theory described in Secs. 4.7 and 4.8 is just what it says—it applies only to thin airfoils at small angles of attack. (Make certain that you understand exactly where in the development of thin airfoil theory these assumptions are made and the reasons for making them.) The advantage of thin airfoil theory is that closed-form expressions are obtained for the aerodynamic coefficients. Moreover, the results compare favorably with experimental data for airfoils of about 12 percent thickness or less. However, the airfoils on many low-speed airplanes are thicker than 12 percent. Moreover, we are frequently interested in high angles of attack, such as occur during takeoff and landing.